# MECHANICS

## Frequency function and damping function in assessment of dynamic processes in mechanical oscillatory systems with symmetry

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*Introduction.* A new approach to the formation of the methodological basis of system analysis in the application to the problems on mechanical oscillatory structure dynamics is considered. The study objective is to develop a method for evaluating properties of the mechanical oscillatory systems with account for viscous friction forces based on frequency functions and damping functions that depend on the so-called coefficient of connection forms, which is the ratio of characteristics of generalized coordinates.

*Materials and Methods.* The graphoanalytical methods used for evaluating the dynamic properties of mechanical oscillatory two-degree-of-freedom systems are based on determining the extreme values of the frequency functions and the damping function, which are determined from the relations between the kinetic, potential energy and the values of the energy dissipation function. Mathematical models are based on Lagrange formalism, matrix methods, and elements of the theory of functions of a complex variable.

*Results.* A method is proposed for constructing frequency functions and damping functions for a class of mechanical oscillatory two-degree-of-freedom systems based on the analytical expressions that reflect features of the ratio of the potential and kinetic energy of the system considering viscous friction forces represented by the dissipative function. General analytical expressions for the frequency function and the damping function are derived. Graphoanalytical analysis of extreme properties of the corresponding frequency functions and damping functions is performed for mechanical vibrational systems with elastic-damping elements with symmetry properties. The results of numerical experiments are presented. A criterion for classifying frequency functions and damping functions based on the topological features of the graphs of the corresponding functions is proposed.

*Discussion and Conclusions.* The developed method for constructing frequency functions and damping functions can be used to display the dynamic features of mechanical oscillatory systems. The proposed matrix method for constructing a frequency-damping function for a two-degree-of-freedom system can be extended to the mechanical vibrational systems considered in different coordinate systems.

*Keywords:* mechanical system, dynamic connections, frequency function, damping function, connectivity of movement, extreme properties, oscillation, viscous friction.

**Introduction.** Considerable attention is paid to the methods of using mechanical vibration systems as calculation schemes in the problems on evaluating the dynamic properties of technical objects operating under intense vibration loads [1–9]. Methods based on extreme properties of the ratio of potential and kinetic energy can be considered as common approaches to evaluating the dynamic properties of mechanical oscillatory systems [10, 11]. Methods based on energy relations have been developed in the use of the frequency function as a function of the coefficient of connection forms of the mechanical system coordinates to evaluate the dynamics of mechanical oscillatory systems disregarding friction forces [12–15].

At the same time, methods for evaluating the dynamic properties of mechanical oscillatory systems, with account for the viscous friction forces based on the frequency function, require detailed representations depending on

the viscous friction value. In particular, this is due to the fact that for systems with aperiodic motion, the concept of oscillation frequency may lose its meaning.

The proposed work is devoted to the creation of a method for evaluating the properties of mechanical movements based on the development of the concept of the frequency function, when an additional damping function reflecting the features considering viscous friction forces depending on the coefficients of connection forms, is introduced.

**Materials and Methods.** Free motions of a mechanical elastic-dissipative system with concentrated two-degree-of-freedom parameters are considered. The schematic diagram of the system is shown in Fig. 1.
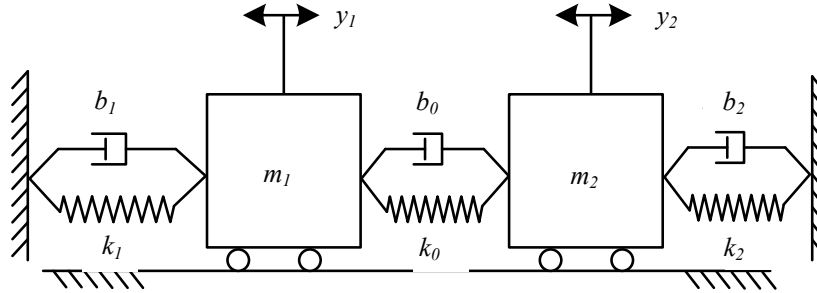


Fig. 1. Mechanical oscillating system with account for viscous friction

Generalized coordinates $y_1$, $y_2$ denote the displacement of mass-inertial elements $m_1$, $m_2$ relative to the static equilibrium position. Kinetic energy $T$, potential energy $\Pi$ and scattering function $F$ have the form:

$$T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2, \tag{1}$$

$$\Pi = \frac{1}{2} k_1 y_1^2 + \frac{1}{2} k_2 y_2^2 + \frac{1}{2} k_0 (y_2 - y_1)^2, \tag{2}$$

$$F = \frac{1}{2} b_1 \dot{y}_1^2 + \frac{1}{2} b_0 (\dot{y}_2 - \dot{y}_1)^2 + \frac{1}{2} b_2 \dot{y}_2^2. \tag{3}$$

The system of Lagrange equations of the second kind has the form:

$$\begin{cases} \dfrac{d}{dt} \dfrac{\partial T}{\partial \dot{y}_1} + \dfrac{\partial \Pi}{\partial y_1} + \dfrac{\partial F}{\partial \dot{y}_1} = 0; \\ \dfrac{d}{dt} \dfrac{\partial T}{\partial \dot{y}_2} + \dfrac{\partial \Pi}{\partial y_2} + \dfrac{\partial F}{\partial \dot{y}_2} = 0. \end{cases} \tag{4}$$

After substituting expressions $T$, $\Pi$, $F$, the system of differential equations (4) takes the form:

$$\begin{cases} m_1 \ddot{y}_1 + (b_0 + b_1)\dot{y}_1 - b_0 \dot{y}_2 + (k_0 + k_1)y_1 - k_0 y_2 = 0; \\ m_2 \ddot{y}_2 + (b_0 + b_2)\dot{y}_2 - b_0 \dot{y}_1 + (k_0 + k_2)y_2 - k_0 y_1 = 0. \end{cases} \tag{5}$$

The forms of free motions of the presented system (5) are generally determined by the eigenvalues of the characteristic equation, taking into account their multiplicity. The case of simple roots is considered. Thus, let the solution $y_1 = y_1(t)$, $y_1 = y_2(t)$ of the system (5) be represented as:

$$\vec{y} = \vec{Y} e^{pt}, \tag{6}$$

where $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is the solution vector, $\vec{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ is the numeric vector, $p = \sigma + j\omega$ is the complex parameter, $t$ is the time variable. It is assumed that the initial conditions are consistent with the type of solution you are looking for (6).

The task is to construct and evaluate the extreme properties of functions that display the characteristics of the proper motions of the system with account for the viscous friction forces.

**Research Results**

**1. Construction of the frequency function and the dissipation function based on the energy ratio.** The system (5) in the notation (6) has the form:

$$\begin{bmatrix} m_1 p^2 + (b_0 + b_1)p + k_0 + k_1 & -b_0 p - k_0 \\ -b_0 p - k_0 & m_2 p^2 + (b_0 + b_2)p + k_0 + k_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = 0. \tag{7}$$

We introduce the notations:

$$A = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \ B = \begin{bmatrix} b_0 + b_1 & -b_0 \\ -b_0 & b_0 + b_2 \end{bmatrix}, \ C = \begin{bmatrix} k_0 + k_1 & -k_0 \\ -k_0 & k_0 + k_2 \end{bmatrix}. \tag{8}$$

Considering (8), the matrix relation (5) takes the form:

$$(p^2 A + pB + C)\vec{Y} = 0. \tag{9}$$

On the basis of the matrix relation (9), various scalar equations can be obtained. With account for their extreme properties, in turn, the properties of the solution $p$ of the equation (9) can be determined. In particular, scalar multiplication of the left and right parts of the equation (9) by vector $\vec{Y}$ results in the scalar expression:

$$p^2 \langle A\vec{Y}, \vec{Y} \rangle + p \langle B\vec{Y}, \vec{Y} \rangle + \langle C\vec{Y}, \vec{Y} \rangle = 0. \tag{10}$$

Using substitution $p = \sigma + j\omega$, we can write (10) as:

$$(\sigma^2 - \omega^2 + 2j\sigma\omega)\langle A\vec{Y}, \vec{Y} \rangle + (\sigma + j\omega)\langle B\vec{Y}, \vec{Y} \rangle + \langle C\vec{Y}, \vec{Y} \rangle = 0, \tag{11}$$

Let the following relation be fulfilled for the coordinates of vector $\vec{Y}$:

$$Y_2 = \alpha Y_1, \tag{12}$$

where $\alpha$ is the coefficient of the connection form. In this case, vector $\vec{Y}$ can be presented as:

$$\vec{Y} = Y_1 \vec{\alpha}, \tag{13}$$

where $\vec{\alpha} = \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$ is vector defined by the connectivity coefficient $\alpha$. After substituting (13), the expression (11) takes the form:

$$(\sigma^2 - \omega^2 + 2j\sigma\omega)A_\alpha + (\sigma + j\omega)B_\alpha + C_\alpha = 0, \tag{14}$$

where $A_\alpha = \langle A\vec{\alpha}, \vec{\alpha} \rangle$, $B_\alpha = \langle B\vec{\alpha}, \vec{\alpha} \rangle$, $B_\alpha = \langle B\vec{\alpha}, \vec{\alpha} \rangle$ are scalar functions of the coefficient of the connection form $\alpha$. The equation (14) can be presented in the equivalent form:

$$\begin{cases} \omega^2 A_\alpha = \sigma^2 A_\alpha + \sigma B_\alpha + C_\alpha; \\ 2\sigma\omega A_\alpha + \omega B_\alpha = 0. \end{cases} \tag{15}$$

We find the solution to the system (15) with respect to $\sigma, \omega$, as functions $\alpha$. Function $\omega(\alpha)$ is considered as a frequency function, $\sigma(\alpha)$ is the damping function. Features of the solution to the system are determined by the sign of the expression $\sigma^2 A_\alpha + \sigma B_\alpha + C_\alpha$.

*Considering dissipation.* The level of energy dissipation can be characterized by discriminant $B_\alpha{}^2 - 4A_\alpha C_\alpha$. Under the following condition:

$$B_\alpha{}^2 < 4A_\alpha C_\alpha, \tag{16}$$

which is understood as a small amount of friction, the solution (15) can be presented as:

$$\begin{cases} \omega^2 = \dfrac{C_\alpha}{A_\alpha} - \left(\dfrac{B_\alpha}{2A_\alpha}\right)^2; \\ \sigma = -\dfrac{B_\alpha}{2A_\alpha}. \end{cases} \tag{17}$$

It should be noted that when the conditions of smallness of the viscous friction forces (16) are met, the equation is performed:

$$\omega^2 + \sigma^2 = \dfrac{C_\alpha}{A_\alpha}. \tag{18}$$

Under the condition of "large viscous friction forces":

$$B_\alpha{}^2 > 4A_\alpha C_\alpha, \tag{19}$$

the solution (15) can be presented as:

$$\omega = 0, \ (20) \quad \sigma_1(\alpha) = -\dfrac{B_\alpha}{2A_\alpha} - \sqrt{\left(\dfrac{B_\alpha}{2A_\alpha}\right)^2 - \dfrac{C_\alpha}{A_\alpha}}, \ \sigma_2(\alpha) = -\dfrac{B_\alpha}{2A_\alpha} + \sqrt{\left(\dfrac{B_\alpha}{2A_\alpha}\right)^2 - \dfrac{C_\alpha}{A_\alpha}}. \tag{21}$$

The damping function has two components $\sigma_1(\alpha)$, $\sigma_2(\alpha)$, such that:

$$\sigma_1(\alpha) + \sigma_2(\alpha) = -\dfrac{B_\alpha}{A_\alpha}, \ \sigma_1(\alpha) \cdot \sigma_2(\alpha) = \dfrac{C_\alpha}{A_\alpha}. \tag{22}$$

Thus, depending on the level of viscous friction forces, different types of presentation of the frequency function and the damping function are possible. If $B_\alpha^2 - 4A_\alpha C_\alpha < 0$, then the frequency function $\omega^2(\alpha)$ and one component of the damping function $\sigma(\alpha)$ are defined. If $B_\alpha^2 - 4A_\alpha C_\alpha > 0$, then it is assumed that the frequency function $\omega^2(\alpha)$ takes zero values, and the damping function has two different negative components $\sigma_1(\alpha)$, $\sigma_2(\alpha)$.

As for the condition $B_\alpha^2 - 4A_\alpha C_\alpha = 0$, it requires a separate analysis. However, the condition $B_\alpha^2 - 4A_\alpha C_\alpha = 0$ can be interpreted as a boundary between two different modes of motion of a mechanical system.

The presented analytical expressions of the frequency function, the damping function, and the conditions of "small" and "large" viscous friction forces can be detailed when considering specific options of mechanical oscillatory systems obtained on the basis of a two-degree-of-freedom system.

**2. Frequency function and damping function for a mechanical two-degree-of-freedom system.** Parameter options for the mechanical system shown in Fig.1 are considered. It is supposed that a set of boundary parameters separating the modes of motion for small and large forces of viscous friction is determined from the equation:

$$B_\alpha^2 = 4A_\alpha C_\alpha, \tag{23}$$

where:

$$A_\alpha = m_1 + m_2\alpha^2, \tag{24}$$
$$B_\alpha = (b_0 + b_2)\alpha^2 - 2\alpha b_0 + b_0 + b_1, \tag{25}$$
$$C_\alpha = (k_0 + k_2)\alpha^2 - 2\alpha k_0 + k_0 + k_1. \tag{26}$$

The conditions of smallness of viscous friction forces have the form:

$$B_\alpha^2 < 4A_\alpha C_\alpha. \tag{27}$$

In this case, frequency function $\omega^2(\alpha)$ and damping function $\sigma(\alpha)$:

$$\begin{cases} \omega^2(\alpha) = \dfrac{(k_0 + k_2)\alpha^2 - 2\alpha k_0 + k_0 + k_1}{m_1 + m_2\alpha^2} - \left(\dfrac{1}{2}\dfrac{(b_0 + b_2)\alpha^2 - 2\alpha b_0 + b_0 + b_1}{m_1 + m_2\alpha^2}\right)^2 \\[4mm] \sigma(\alpha) = -\dfrac{1}{2}\dfrac{(b_0 + b_2)\alpha^2 - 2\alpha b_0 + b_0 + b_1}{m_1 + m_2\alpha^2} \end{cases}. \tag{28}$$

The conditions of large viscous friction forces have the form:

$$B_\alpha^2 > 4A_\alpha C_\alpha. \tag{29}$$

Under the conditions (29), functions $\omega^2$ and $\sigma(\alpha)$ have the form:

$$\begin{cases} \omega^2 = 0; \\[2mm] \sigma_1(\alpha) = -\dfrac{1}{2}\dfrac{(b_0 + b_2)\alpha^2 - 2\alpha b_0 + b_0 + b_1}{m_1 + m_2\alpha^2} - \\[2mm] \quad - \sqrt{\left(\dfrac{1}{2}\dfrac{(b_0 + b_2)\alpha^2 - 2\alpha b_0 + b_0 + b_1}{m_1 + m_2\alpha^2}\right)^2 - \dfrac{(k_0 + k_2)\alpha^2 - 2\alpha k_0 + k_0 + k_1}{m_1 + m_2\alpha^2}}; \\[2mm] \sigma_2(\alpha) = -\dfrac{1}{2}\dfrac{(b_0 + b_2)\alpha^2 - 2\alpha b_0 + b_0 + b_1}{m_1 + m_2\alpha^2} + \\[2mm] \quad + \sqrt{\left(\dfrac{1}{2}\dfrac{(b_0 + b_2)\alpha^2 - 2\alpha b_0 + b_0 + b_1}{m_1 + m_2\alpha^2}\right)^2 - \dfrac{(k_0 + k_2)\alpha^2 - 2\alpha k_0 + k_0 + k_1}{m_1 + m_2\alpha^2}}. \end{cases} \tag{30}$$

The presented expressions reflect motions in the form of exponential decrease in the absence of fluctuations.

**3. Features of frequency functions and damping functions for symmetric mechanical oscillatory systems.** We consider a mechanical oscillating system with elastic damping elements, whose parameter values are imposed by symmetry conditions in the form $b_1 = b_2 = b_0 = b$, $k_1 = k_2 = k_0 = k$. The schematic diagram is shown in Fig. 2.
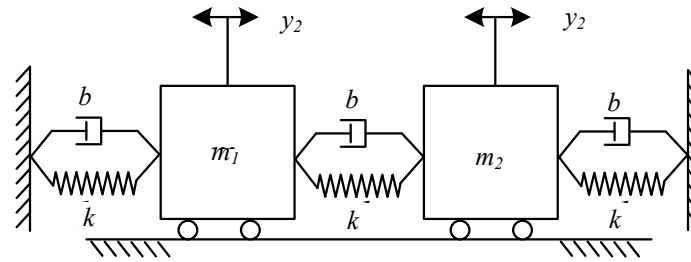
Fig. 2. "Symmetrical" mechanical system

The system of differential equations (5) has the form:

$$\begin{cases} m\ddot{y}_1 + 2b\dot{y}_1 - b\dot{y}_2 + 2ky_1 - ky_2 = 0; \\ m\ddot{y}_2 + 2b\dot{y}_2 - b\dot{y}_1 + 2ky_2 - ky_1 = 0. \end{cases} \tag{31}$$

Functions $A_\alpha$, $B_\alpha$, $C_\alpha$ can be presented by the expressions:

$$A_\alpha = m_1 + m_2\alpha^2, \tag{32}$$

$$B_\alpha = 2b(\alpha^2 - \alpha + 1), \tag{33}$$

$$C_\alpha = 2k(\alpha^2 - \alpha + 1). \tag{34}$$

On the basis of the presented components, frequency function and damping function can be constructed, and the conditions for the smallness of the viscous friction forces can be formulated.

*Accounting for viscous friction forces.* The condition of smallness of the friction forces can be presented from the inequality:

$$\left(\frac{B_\alpha}{2A_\alpha}\right)^2 < \frac{C_\alpha}{A_\alpha}. \tag{35}$$

After substituting the functions (32) – (34), the condition of smallness of the friction forces (35) can be written as:

$$\gamma_0 < M(\alpha), \tag{36}$$

where $\gamma_0 = \dfrac{b^2}{4k}$ is a generalized viscoelastic parameter, $M(\alpha) = \dfrac{1}{2} \cdot \dfrac{m_1 + m_2\alpha^2}{\alpha^2 - \alpha + 1}$ is a generalized mass-inertia coefficient that depends on the shape coefficient $\alpha$. The graph of function $M_\alpha$ for each fixed $\gamma_0$ defines a set of values for $\alpha$, at which the condition of smallness of the friction forces is satisfied.

As an example, Fig. 3 shows a graph of the parameterizing function $M(\alpha)$. Function $M(\alpha)$ has global minimum $M_1$ and maximum $M_2$ at $\alpha \to \infty$ $M(\alpha) \to \dfrac{m_2}{2}$.
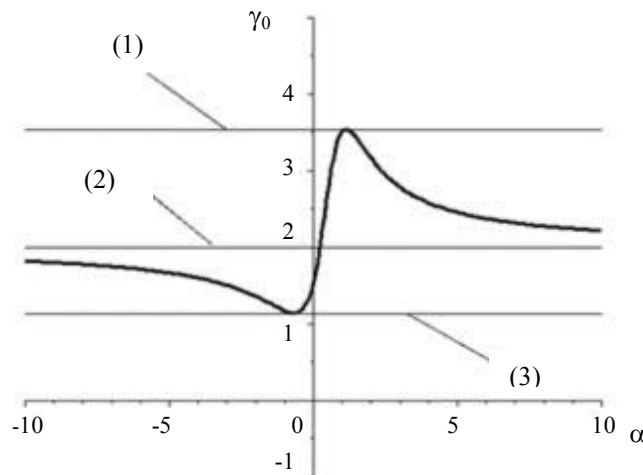


Fig. 3. Parameterizing function $M(\alpha)$: 1 — global maximum level $M_2$,

3 — global minimum level $M_1$, 2 — level of horizontal asymptote $\dfrac{m_2}{2}$

The change in parameter $\gamma_0$ in the intervals $(0, M_1), (M_1, \frac{m_2}{2}), (\frac{m_2}{2}, M_2), (M_2, \infty)$ determines the characteristic intervals of the shape coefficient $\alpha$, under which the conditions for low friction forces are met.

For low friction forces, frequency function and damping function have the form:

$$\begin{cases} \omega^2(\alpha) = \dfrac{2k(\alpha^2 - \alpha + 1)}{m_1 + m_2\alpha^2} - \left(\dfrac{b(\alpha^2 - \alpha + 1)}{m_1 + m_2\alpha^2}\right)^2; \\ \sigma(\alpha) = -\dfrac{b(\alpha^2 - \alpha + 1)}{m_1 + m_2\alpha^2}. \end{cases} \qquad (37)$$

For high friction forces, at which the aperiodic motion of the system is realized, the frequency function is zero, and the damping function has two components:

$$\begin{cases} \omega^2 = 0; \\ \sigma_1(\alpha) = -\dfrac{b(\alpha^2 - \alpha + 1)}{m_1 + m_2\alpha^2} - \sqrt{\left(\dfrac{b(\alpha^2 - \alpha + 1)}{m_1 + m_2\alpha^2}\right)^2 - \dfrac{2k(\alpha^2 - \alpha + 1)}{m_1 + m_2\alpha^2}}; \\ \sigma_2(\alpha) = -\dfrac{b(\alpha^2 - \alpha + 1)}{m_1 + m_2\alpha^2} + \sqrt{\left(\dfrac{b(\alpha^2 - \alpha + 1)}{m_1 + m_2\alpha^2}\right)^2 - \dfrac{2k(\alpha^2 - \alpha + 1)}{m_1 + m_2\alpha^2}}. \end{cases} \qquad (38)$$

On the basis of analytical representations of the frequency function and the damping function, characteristic variants and features of the extreme properties of the corresponding functions can be determined, taking into account the viscous friction forces.

**Discussion and Conclusion.** Of interest are the characteristic variants of frequency functions and damping functions depending on the conditions of low viscous friction forces. Variants of value $\gamma_0$, that determine the characteristic intervals of the shape coefficient $\alpha$, under which the conditions of smallness of the viscous friction forces are met, are considered.

1. Let $\gamma_0 \in (0, M_1)$. The example $\gamma_0 \approx 0.1$ is considered. In this case, the conditions for low friction forces for any shape coefficient $\alpha \in (-\infty..\infty)$ are met. Fig. 4 and 5 show the frequency function $\omega^2(\alpha)$ and damping function $\sigma(\alpha)$ for the mechanical elastic-dissipative system with parameters $b = 1$, $m_1 = 3$, $m_2 = 4$, $k = 3$.
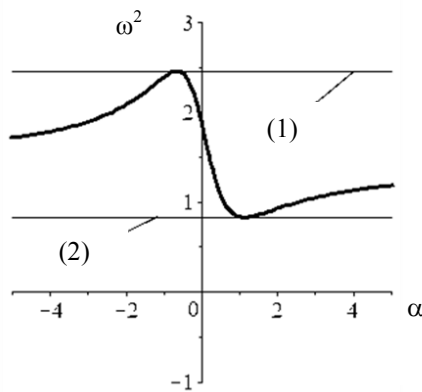


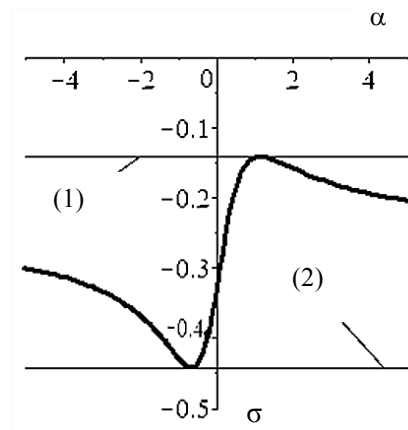Fig. 4. Frequency functions $\omega^2(\alpha)$:
(1) and (2) are extreme levels

Fig. 5. Damping function $\sigma(\alpha)$:
(1) and (2) are extreme levels

Roots of the equation that is equal to zero of the corresponding determinant

$$\left| Ap^2 + Bp + C \right| = 0, \qquad (39)$$

are $p_i = \omega_i + j\sigma_i$, $i = 1..4$, where $\omega_1 \approx 0.91$; $\omega_2 \approx 1.56$; $\omega_3 \approx -1.56$; $\omega_4 \approx -0.91$; $\sigma_1 \approx -0.14$; $\sigma_2 \approx -0.44$; $\sigma_3 \approx -0.44$;

In Fig. 4, the frequency function reaches extreme values equal to the squares of the frequencies $\omega_2^2 \approx 2.46$ and $\omega_1^2 \approx 0.82$. In Fig. 5, the damping function reaches extreme values that are $\sigma_3 \approx -0.44$ and $\sigma_4 \approx -0.14$. The

frequency function and the damping function reach their extreme values when the form coefficients are $\alpha_1^* = -0,65$ and $\alpha_2^* = 1.15$.

2. Let $\gamma_0 \in (M_1, \dfrac{m_2}{2})$. We consider a mechanical system with parameters $b = 4$, $\gamma_0 \approx 1.33$. Fig. 6 and 7 show the corresponding frequency function and damping function. The set of coefficients of forms for which the condition of low friction forces is satisfied is: $(-\infty, \alpha_1) \cup (\alpha_2, \infty)$, where $\alpha_i$ are the roots of the equation $M(\alpha) = \gamma_0$. For parameters $b = 4$, $m_1 = 3$, $m_2 = 4$, $k = 3$ the roots of the characteristic equation (39) have real parts representing the dissipation coefficients, $\sigma_1 \approx -0.57$; $\sigma_2 \approx -1.08$; $\sigma_3 \approx -2.46$; $\sigma_4 \approx -0.57$ and imaginary parts representing the frequencies $\omega_1 \approx 0.73$; $\omega_2 \approx 0$; $\omega_3 \approx 0$; $\omega_4 \approx -0.73$. In Fig. 6, the frequency function has a local minimum $\omega_4^2 \approx 0.53$ in the interval $(-\infty, \alpha_1) \cup (\alpha_2, \infty)$.
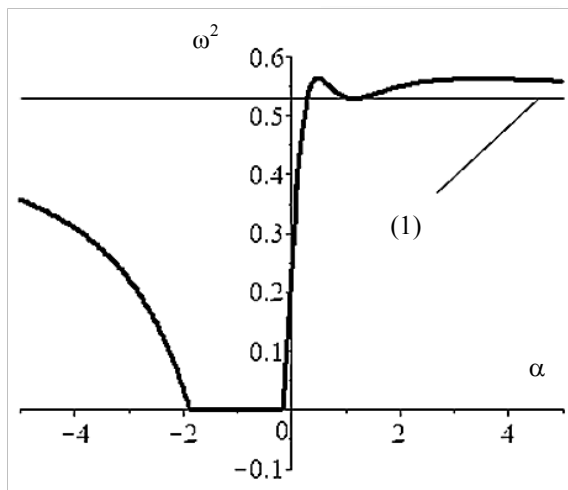
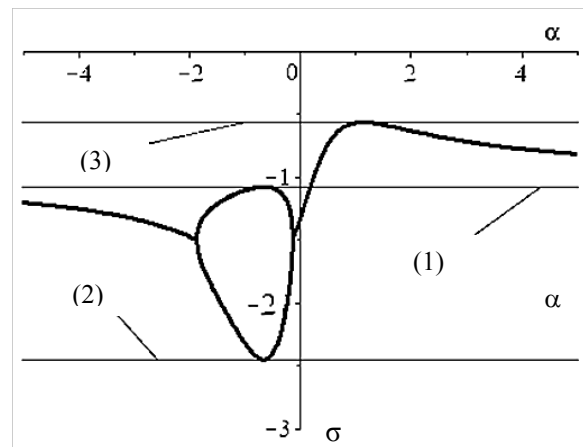Fig. 6. Frequency function:
(1) is extreme level at point $\alpha_2^*$

Fig. 7. Damping function:
(3) is extreme level at point $\alpha_2^*$; (1), (2) are extreme levels at point $\alpha_1^*$ on the two-valued interval

On the interval $(\alpha_1, \alpha_2)$, the frequency function is zero. In turn, in Fig. 7, the damping function in the interval $(\alpha_1, \alpha_2)$ is double-valued and reaches simultaneously two extreme values $\sigma_2 \approx -1.08$ and $\sigma_3 \approx -2.46$ at point $\alpha_2^*$. In the domain $(-\infty, \alpha_1) \cup (\alpha_2, \infty)$, the damping function is single-valued and has one local extremum $\sigma_1 \approx -0.57$ at point $\alpha_2^*$.

3. Let $\gamma_0 \in (\dfrac{m_2}{2}, M_2)$. We consider a mechanical system with parameters $b = 6$; $m_1 = 3$; $m_2 = 4$; $k = 3$. The roots of the characteristic equation (39) have real $\sigma_1 \approx -0.85$; $\sigma_2 \approx -0.56$; $\sigma_3 \approx -4.74$; $\sigma_4 \approx -0.85$ and imaginary parts $\omega_1 \approx 0.36$; $\omega_2 \approx 0$; $\omega_3 \approx 0$; $\omega_4 \approx -0.36$. The conditions for low friction forces are met in the interval $(\alpha_1, \alpha_2)$, where $\alpha_1 \approx 0.63$; $\alpha_2 \approx 2.37$. In Fig. 8, the corresponding frequency function is positive only on the interval $(\alpha_1, \alpha_2)$. The local extremum of the frequency function is $\omega_4^2 \approx 0.13$. Outside the interval $(\alpha_1, \alpha_2)$, the frequency function is zero.

In Fig. 9, the damping function is double-valued in the interval $(-\infty, \alpha_1)$ and reaches simultaneously two extreme values at point $\alpha_1^*$, which are $\sigma_3 \approx -4.74$ and $\sigma_2 \approx -0.56$. In the interval $(\alpha_2, \infty)$ the damping function is also double-valued. In the interval $(\alpha_1, \alpha_2)$, the damping function is single-valued and has one local extremum $\sigma_1 \approx -0.85$ at point $\alpha_2^*$.
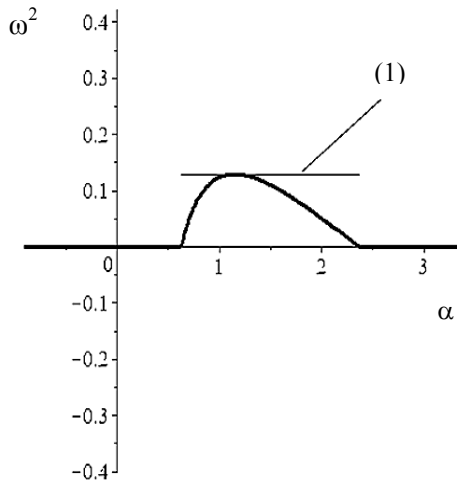
Fig. 8. Frequency function:
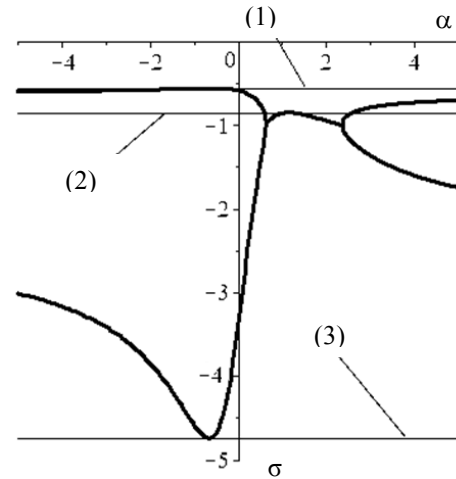(1) is extreme level at point $\alpha_2^*$



Fig. 9. Damping function: (1), (3) are extreme levels at point $\alpha_1^*$
on the two-valued interval, (2) is extreme level at point $\alpha_2^*$

4. Let $\gamma_0 \in (M_2, \infty)$. For parameters $b = 6.52$; $m_1 = 3$; $m_2 = 4$; $k = 3$, the characteristic equation (39) has only real roots $\sigma_1 \approx -0.50$; $\sigma_2 \approx -0.88$; $\sigma_3 \approx -0.96$; $\sigma_4 \approx -5.26$. The corresponding frequency function and damping function are shown in Fig. 10 and 11. The interval of fulfillment of the conditions of smallness of the friction forces degenerates into an empty set.
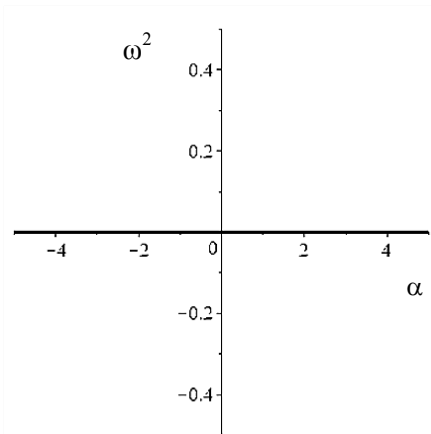


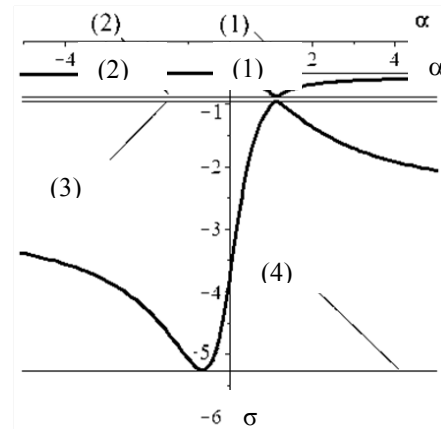Fig. 10. Frequency function:
case of degeneracy



Fig. 11. Damping function, formed by two branches: (1), (2) are extreme levels of the "upper" branch at points $\alpha_1^*$ and $\alpha_2^*$; (3), (4) are extreme levels of the "lower" branch at points $\alpha_1^*$ and $\alpha_2^*$

In Fig. 10, the frequency function is zero on the whole number axis. In Fig. 11, the corresponding damping function is double-valued on the entire numeric axis and has 4 local extrema $\sigma_1 \approx -0.50$; $\sigma_2 \approx -0.88$; $\sigma_3 \approx -0.96$; $\sigma_4 \approx -5.26$ at points $\alpha_1^*$ and $\alpha_2^*$.
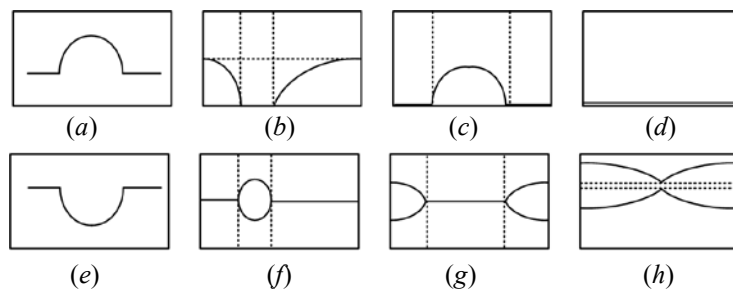


Fig. 12. Pictograms of topological features of frequency function and damping function graphs: (a)-(d) are pictograms associated with graphs of frequency functions shown in Fig. 4, 6, 8, 10, respectively; (e)-(h) are pictograms associated with damping functions in Fig. 5, 7, 9, 11, respectively

The pictograms shown in Fig. 12, compared to the function graphs in Fig. 4–11, reflect a number of topological features of the frequency function and damping function graphs. These features include the shape of the graph as a single curve, the presence of bifurcation points on the graph of one curve into two, the presence of two nonintersecting curves or a "ring".

Thus, the achieved extreme values of the frequency function and the damping function are related to the dynamic characteristics of the mechanical oscillatory system taking into account the friction forces. In particular, the extreme values of the constructed frequency function and damping function are related to the proper frequencies and dissipative coefficients of damped oscillations. In this case, the issue on the existence of extreme values of the frequency function and the damping function that do not coincide with the squares of the proper oscillation frequencies requires additional consideration. At the same time, it can be assumed that the forms of frequency functions and damping functions that determine the modes of free motions of mechanical oscillatory systems with friction are of interest for evaluating a wider range of dynamic properties.

In terms of practical implementation of the possible control of oscillatory modes of mechanical systems based on the connectivity coefficient, there are no fundamental obstacles. For example, possible dynamic state control systems may include sensors of vibration amplitudes at control points of a vibrating process machine or vehicle. However, the construction of such systems requires detailed consideration of a wide range of features related to the technical object.

In conclusion, the following points can be noted as the conclusions of the presented studies.

1. For the mechanical vibrating system considering forces of viscous friction, a method of constructing frequency function and damping function that are dependent on the connection form coefficient of the free motion coordinates, is developed. It is shown that the set of extreme values of the frequency function and the damping function displays the proper characteristics of an elastic-dissipative mechanical oscillatory system.

2. It is shown that the frequency function and damping functions for a mechanical oscillatory two-degree-of-freedom system with account for viscous friction, can be represented in two variants determined by the conditions for the value of the viscous friction forces for a fixed connection form coefficient; for the conditions of low viscous friction forces, the values of the frequency function take positive values, and the damping function has one negative component; if the conditions of high viscous friction forces are met, the frequency function takes zero values, and the damping function has two negative components.

3. A method is proposed for constructing possible variants of frequency functions and damping functions for various values of system parameters based on a parameterizing function that allows determining the regions of values of the connection form coefficient in which the condition of smallness of viscous friction forces is met. A criterion for classifying frequency functions and damping functions depending on the topological features of their graphs is proposed.

4. The matrix method for constructing the frequency-damping function for a two-degree-of-freedom system can be extended to mechanical oscillatory systems considered in different coordinate systems.

5. As a physical interpretation of the connection coefficient used in the frequency function and the damping function, we can consider the linkage in the form of a gear ratio expressed from the ratio of the amplitudes of the partial block coordinate oscillations. The ratio under consideration, along with the static state, can be determined for steady-state and damped oscillation modes. Thus, a concept is developed in which the starting point for the analysis of a mechanical system is the linkage.

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